Vectors (Trigonometry Explanation)

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CONCEPT 1

Vectors (Trigonometry Explanation)

Learning Objectives

- Understand directed line segments, equal vectors, and absolute value in relation to vectors.
- Perform vector addition and subtraction.
- Find the resultant vector of two displacements.

In previous examples, we could simply use triangles to represent direction and distance. In real-life, there are typically other factors involved, such as the speed of the object (that is moving in the given direction and distance) and wind. We need another tool to represent not only direction but also magnitude (length) or force. This is why we need vectors. Vectors capture the interactions of real world velocities, forces and distance changes.

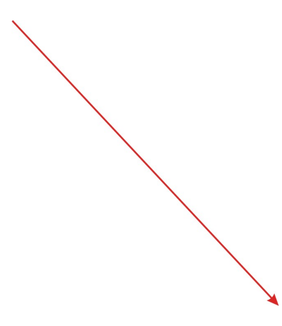
Any application in which direction is specified requires the use of vectors. A **vector** is any quantity having **direction** and **magnitude**. Vectors are very common in science, particularly physics, engineering, electronics, and chemistry in which one must consider an object's motion (either velocity or acceleration) and the direction of that motion.

In this section, we will look at how and when to use vectors. We will also explore vector addition, subtraction, and the resultant of two displacements. In addition we will look at real-world problems and application involving vectors.

Directed Line Segments, Equal Vectors, and Absolute Value

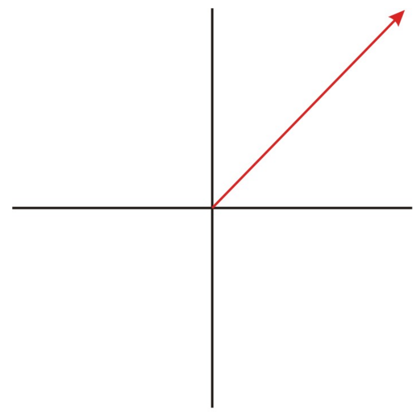
A vector is represented diagrammatically by a directed line segment or arrow. A **directed line segment** has both **magnitude** and **direction**. **Magnitude** refers to the length of the directed line segment and is usually based on a scale. The vector quantity represented, such as influence of the wind or water current may be completely invisible.

A 25 mph wind is blowing from the northwest. If $1\ cm = 5\ mph$, then the vector would look like this:



An object affected by this wind would travel in a southeast direction at 25 mph.

A vector is said to be in **standard position** if its **initial point** is at the origin. The initial point is where the vector begins and the **terminal point** is where it ends. The axes are arbitrary. They just give a place to draw the vector.



vector in standard position

If we know the coordinates of a vector's initial point and terminal point, we can use these coordinates to find the magnitude and direction of the vector.

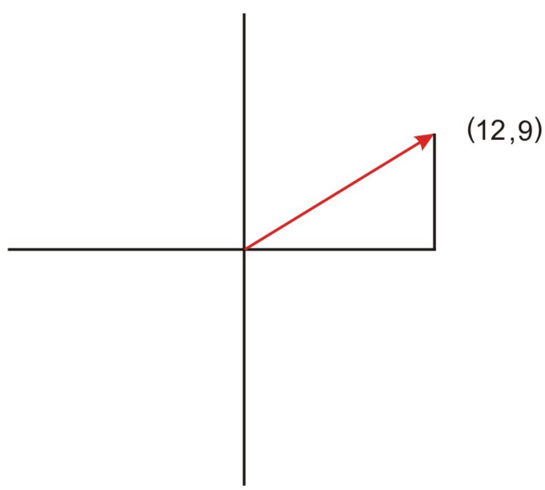
All vectors have **magnitude**. This measures the total distance moved, total velocity, force or acceleration. "Distance" here applies to the magnitude of the vector even though the vector is a measure of velocity, force, or acceleration. In order to find the magnitude of a vector, we use the distance formula. A vector can have a negative magnitude. A force acting on a block pushing it at 20 lbs north can be also written as vector acting on the block from the south with a magnitude of -20 lbs. Such negative magnitudes can be confusing; making a diagram helps. The -20 lbs south can be re-written as +20 lbs north without changing the vector. Magnitude is also called the **absolute value** of a vector.

Example 1: If we know the coordinates of the initial point and the terminal point, we can find the magnitude by using the distance formula. Initial point (0,0) and terminal point (3,5).

Solution: $|\vec{v}| = \sqrt{(3-0)^2+(5-0)^2} = \sqrt{9+25} = 5.8$ The magnitude of$\vec{v}$ is 5.8.

If we don't know the coordinates of the vector, we must use a ruler and the given scale to find the magnitude. Also notice the notation of a vector, which is usually a lower case letter (typically u, v, or w) in italics, with an arrow over it, which indicates direction. If a vector is in standard position, we can use trigonometric ratios such as sine, cosine and tangent to find the **direction** of that vector.

Example 2: If a vector is in standard position and its terminal point has coordinates of (12, 9) what is the direction?



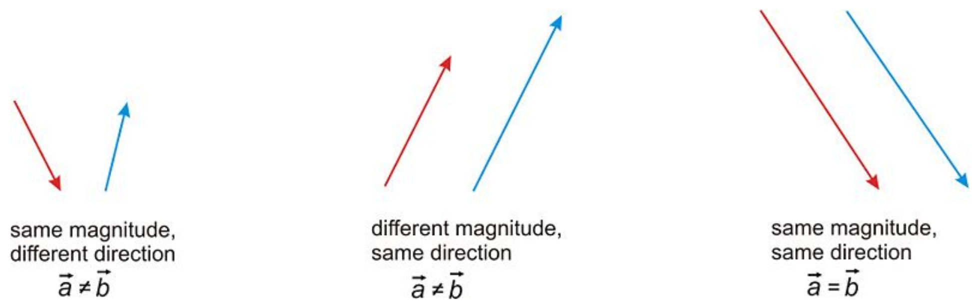
Solution: The horizontal distance is 12 while the vertical distance is 9. We can use the tangent function since we know the opposite and adjacent sides of our triangle.

$$\tan\theta = \frac{9}{12}$$
$$\tan^{-1}\frac{9}{12} = 36.9^\circ$$

So, the direction of the vector is 36.9°.

If the vector isn't in standard position and we don't know the coordinates of the terminal point, we must a protractor to find the direction.

Two vectors are **equal** if they have the same magnitude and direction. Look at the figures below for a visual understanding of **equal vectors**.



Example 3: Determine if the two vectors are equal.

$\vec{a}$ is in standard position with terminal point (-4, 12)

$\vec{b}$ has an initial point of (7, -6) and terminal point (3, 6)

Solution: You need to determine if both the magnitude and the direction are the same.

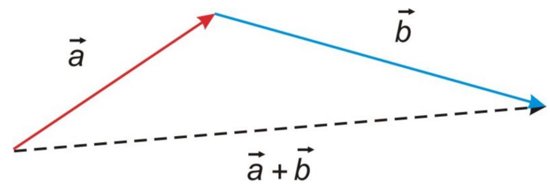
$$\begin{aligned}
\text{Magnitude :} \quad & |\vec{a}| = \sqrt{(0-(-4))^2+(0-12)^2} = \sqrt{16+144} = \sqrt{160} = 4\sqrt{10} \\
& |\vec{b}| = \sqrt{(7-3)^2+(-6-6)^2} = \sqrt{16+144} = \sqrt{160} = 4\sqrt{10} \\
\text{Direction :} \quad & \vec{a} \rightarrow \tan\theta = \frac{12}{-4} \rightarrow \theta = 108.43^\circ \\
& \vec{b} \rightarrow \tan\theta = \frac{-6-6}{7-3} = \frac{-12}{4} \rightarrow \theta = 108.43^\circ
\end{aligned}$$

Because the magnitude and the direction are the same, we can conclude that the two vectors are equal.

Vector Addition

The sum of two or more vectors is called the **resultant** of the vectors. There are two methods we can use to find the resultant: the triangle method and the parallelogram method.

The Triangle Method: To use the triangle method, we draw the vectors one after another and place the initial point of the second vector at the terminal point of the first vector. Then, we draw the resultant vector from the initial point of the first vector to the terminal point of the second vector. *This method is also referred to as the tip-to-tail method.*



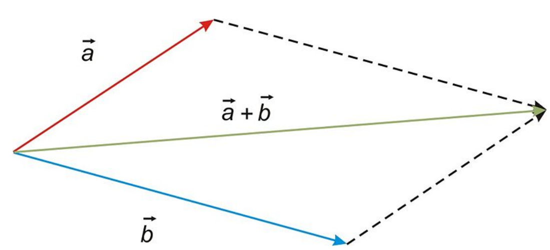
To find the sum of the resultant vector we would use a ruler and a protractor to find the magnitude and direction.

The resultant vector can be much longer than either $\vec{a}$ or $\vec{b}$, or it can be shorter. Below are some more examples of the triangle method.

Example 4:

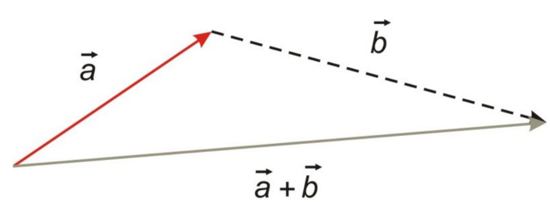
Example 5:

The Parallelogram Method: Another method we could use is the parallelogram method. To use the parallelogram method, we draw the vectors so that their initial points meet. Then, we draw in lines to form a parallelogram. The resultant is the diagonal from the initial point to the opposite vertex of the parallelogram. *It is important to note that we cannot use the parallelogram method to find the sum of a vector and itself.*



To find the sum of the resultant vector, we would again use a ruler and a protractor to find the magnitude and direction.

If you look closely, you'll notice that the parallelogram method is really a version of the triangle or tip-to-tail method. If you look at the top portion of the figure above, you can see that one side of our parallelogram is really vector *b* translated.

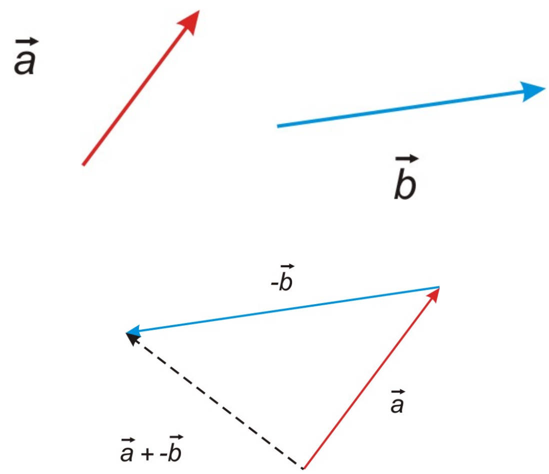


Vector Subtraction

As you know from Algebra, $A - B = A + (-B)$. When we think of vector subtraction, we must think about it in terms of adding a negative vector. A **negative** vector is the same magnitude of the original vector, but its direction is opposite.

In order to subtract two vectors, we can use either the triangle method or the parallelogram method from above. The only difference is that instead of adding vectors A and B, we will be adding A and $-B$.

Example 6: Using the triangle method for subtraction.

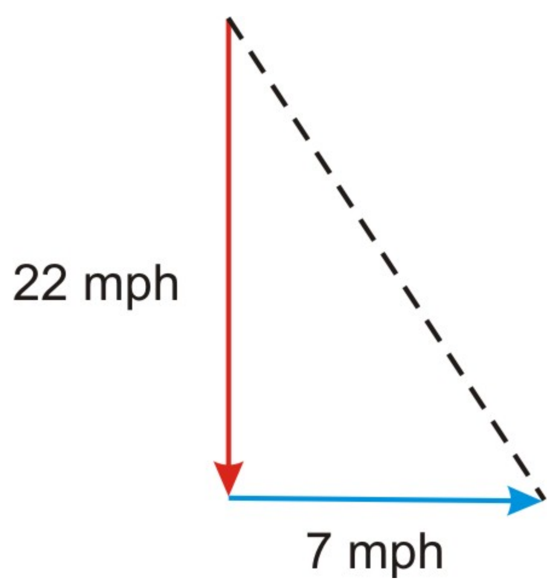


Resultant of Two Displacements

We can use vectors to find direction, velocity, and force of moving objects. In this section we will look at a few applications where we will use resultants of vectors to find speed, direction, and other quantities. A displacement is a distance considered as a vector. If one is 10 ft away from a point, then any point at a radius of 10 ft from that point satisfies the condition. If one is 28 degrees to the east of north, then only one point satisfies this.

Example 7: A cruise ship is traveling south at 22 mph. A wind is also blowing the ship eastward at 7 mph. What speed is the ship traveling at and in what direction is it moving?

Solution: In order to find the direction and the speed the boat is traveling, we must find the resultant of the two vectors representing 22 mph south and 7 mph east. Since these two vectors form a right angle, we can use the Pythagorean Theorem and trigonometric ratios to find the magnitude and direction of the resultant vector.



First, we will find the speed.

$$\begin{aligned} 22^2 + 7^2 &= x^2 \\ 533 &= x^2 \\ 23.1 &= x \end{aligned}$$

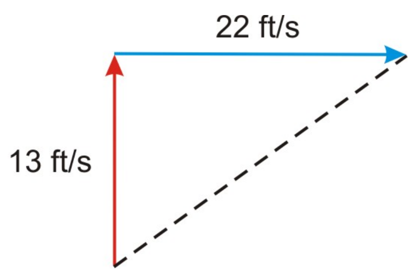
The ship is traveling at a speed of 23.1mph.

To find the direction, we will use tangent, since we know the opposite and adjacent sides of our triangle.

$$\tan\theta = \frac{7}{22}$$
$$\tan^{-1}\frac{7}{22} = 17.7^\circ$$

The ship's direction is $S17.7^\circ E$.

Example 8: A hot air balloon is rising at a rate of 13 ft/sec, while a wind is blowing at a rate of 22 ft/sec. Find the speed at which the balloon is traveling as well as the angle its angle of elevation.



First, we will find the speed at which our balloon is rising. Since we have a right triangle, we can use the Pythagorean Theorem to find calculate the magnitude of the resultant.

$$x^2 = 13^2 + 22^2$$
$$x^2 = 653$$
$$x = 25.6\ ft/\text{sec}$$

The balloon is traveling at rate of 25.6 feet per second.

To find the angle of elevation of the balloon, we need to find the angle it makes with the horizontal. We will find the angle A in the triangle and then we will subtract it from 90°.

$$\tan A = \frac{22}{13}$$
$$A = \tan^{-1}\frac{22}{13}$$
$$A = 59.4^\circ$$

Angle with the horizontal $= 90 - 59.4 = 30.6^\circ$.

The balloon has an angle of elevation of 30.6°.

Example 9: Continuing on with the previous example, find:

a. How far from the lift off point is the balloon in 2 hours? Assume constant rise and constant wind speed. (this is *total displacement*)

b. How far must the support crew travel on the ground to get under the balloon? (*horizontal displacement*)

c. If the balloon stops rising after 2 hours and floats for another 2 hours, how far from the initial point is it at the end of the 4 hours? How far away does the crew have to go to be under the balloon when it lands?

Solution:

a. After two hours, the balloon will be 184,320 feet from the lift off point (25.6 ft/sec multiplied by 7200 seconds in two hours).

b. After two hours, the horizontal displacement will be 158,400 feet (22ft/sec multiplied by 7200 seconds in two hours).

c. After two hours, the balloon will have risen 93,600 feet. After an additional two hours of floating (horizontally only) in the 22ft/sec wind, the balloon will have traveled 316,800 feet horizontally (22ft/second times 14,400 seconds in four hours).

We must recalculate our resultant vector using Pythagorean Theorem.

$$x = \sqrt{93600^2 + 316800^2} = 330338\ ft.$$

The balloon is 330,338 feet from its initial point. The crew will have to travel 316,800 feet or 90 miles (horizontal displacement) to be under the balloon when it lands.

Points to Consider

- Is it possible to find the magnitude and direction of resultants without using a protractor and ruler and without using right triangles?
- How can we use the Law of Cosines and the Law of Sines to help us find magnitude and direction of resultants?

Review Questions

1. Vectors $\vec{m}$ and $\vec{n}$ are perpendicular. Make a diagram of each addition, find the magnitude and direction (with respect to $\vec{m}$ and $\vec{n}$) of their resultant if:
   a. $|\vec{m}| = 29.8 |\vec{n}| = 37.7$
   b. $|\vec{m}| = 2.8 |\vec{n}| = 5.4$
   c. $|\vec{m}| = 11.9 |\vec{n}| = 9.4$
2. For $\vec{a}, \vec{b}, \vec{c}$, and $\vec{d}$ below, make a diagram of each addition or subtraction. $|\vec{a}| = 6cm$, direction $= 45^\circ |\vec{b}| = 3.2cm$, direction $= 30^\circ |\vec{c}| = 1.3cm$, direction $= 110^\circ |\vec{d}| = 4.8cm$, direction $= 80^\circ$
   a. $\vec{a} + \vec{b}$
   b. $\vec{a} + \vec{d}$
   c. $\vec{c} + \vec{d}$
   d. $\vec{a} - \vec{d}$
   e. $\vec{b} - \vec{a}$
   f. $\vec{d} - \vec{c}$
3. Does $|\vec{a} + \vec{b}| = |\vec{a}| + |\vec{b}|$? Explain your answer.
4. A plane is traveling north at a speed of 225 mph while an easterly wind is blowing the plane west at 18 mph. What is the direction and the speed of the plane?
5. Two workers are pulling on ropes attached to a tree stump. One worker is pulling the stump east with 330 Newtons of forces while the second working is pulling the stump north with 410 Newtons of force. Find the magnitude and direction of the resultant force on the tree stump.
6. Assume $\vec{a}$is in standard position. For each terminal point is given, find the magnitude and direction of each vector.
   a. (12, 18)
   b. (-3, 6)
7. Given the initial and terminal coordinates of $\vec{a}$, find the magnitude and direction.

a. initial (2, 4) terminal (8, 6)
b. initial (5, -2) terminal (3, 1)

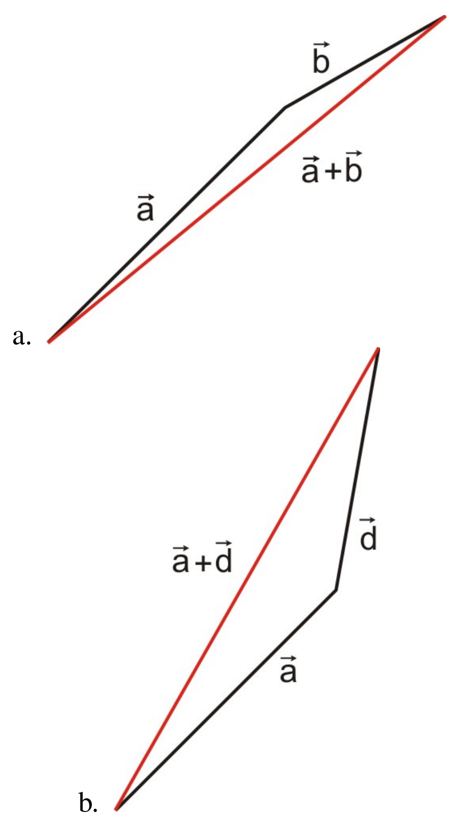
8. The magnitudes of vectors $\vec{a}$ and $\vec{b}$ are given, along with the angle they make with each other, theta, when positioned tip-to-tail. Find the magnitude of the resultant and the angle it makes with a.

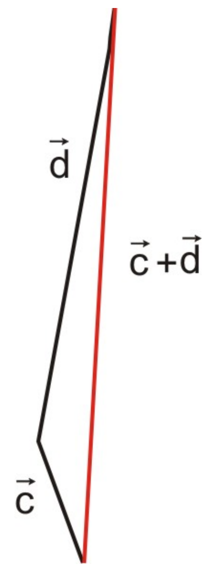
a. $|\vec{a}| = 31, |\vec{b}| = 31, \theta = 132^\circ$
b. $|\vec{a}| = 29, |\vec{b}| = 44, \theta = 26^\circ$

Review Answers

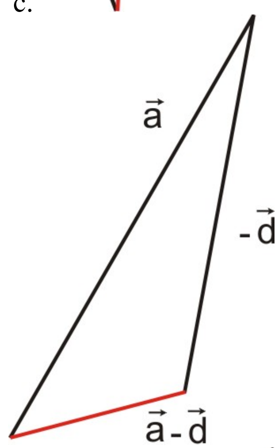
1. For each problem below, use the Pythagorean Theorem to find the magnitude and $\tan\theta = \frac{|\vec{n}|}{|\vec{m}|}$

a. magnitude $= 48.1$, direction $= 51.7^\circ$
b. magnitude $= 6.1$, direction $= 62.6^\circ$
c. magnitude $= 15.2$, direction $= 38.3^\circ$

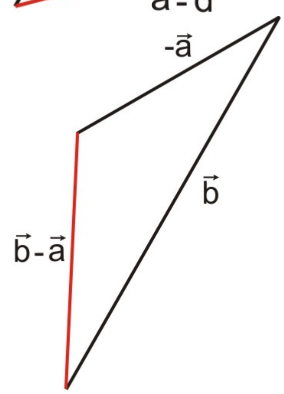




c.

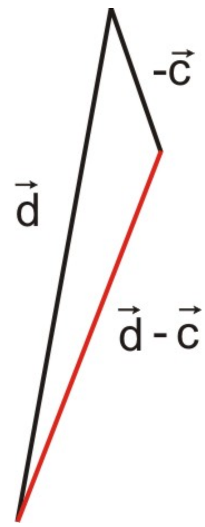


d.



e.

f.



2. When two vectors are summed, the magnitude of the resulting vector is almost always different than the sum of the magnitudes of the two initial vectors. The only times that $|\vec{a}+\vec{b}| = |\vec{a}|+|\vec{b}|$ would be true is when 1) the magnitude of at least one of the two vectors to be added is zero, or 2) both of the vectors to be added have the same direction.
3. Speed (magnitude): $\sqrt{18^2+225^2} = 225.7$ and its direction is $\tan\theta = \frac{18}{225} = N4.6^\circ W$.
4. The magnitude is $\sqrt{330^2+410^2} = 526.3$ Newtons and the direction is $\tan^{-1}\left(\frac{410}{330}\right) = E51.2^\circ N$.

   a. $|\vec{a}| = \sqrt{12^2+18^2} = 21.6$, direction $= \tan^{-1}\left(\frac{18}{12}\right) = 56.3^\circ$.
   b. $|\vec{a}| = \sqrt{(-3)^2+6^2} = 6.7$, direction $= \tan^{-1}\left(\frac{6}{-3}\right) = 116.6^\circ$.

   a. $|\vec{a}| = \sqrt{(2-8)^2+(4-6)^2} = 6.3$, direction $= \tan^{-1}\left(\frac{4-6}{2-8}\right) = 18.4^\circ$.
   b. $|\vec{a}| = \sqrt{(5-3)^2+(-2-1)^2} = 3.6$, direction $= \tan^{-1}\left(\frac{-2-1}{5-3}\right) = 123.7^\circ$. Note that when you use your calculator to solve for $\tan^{-1}(\frac{-2-1}{5-3})$, you will get -56.3°. The calculator produces this answer because the range of the calculator's $y = \tan^{-1} x$ function is limited to $-90^\circ < y < 90^\circ$. You need to sketch a draft of the vector to see that its direction when placed in standard position is into the second quadrant (and not the fourth quadrant), and so the correct angle is calculated by moving the angle into the second quadrant through the equation $-56.3^\circ + 180^\circ = 123.7^\circ$.

5. In both a and b, we have the SAS case, so you can do the Law of Cosines, followed by the Law of Sines.

   a. $(\vec{a}+\vec{b})^2 = 31^2+31^2-2(31)(31)\cos 132, \vec{a}+\vec{b} = 56.6, \frac{\sin 132}{56.6} = \frac{\sin x}{31}, x = 24^\circ$
   b. $(\vec{a}+\vec{b})^2 = 29^2+44^2-2(29)(44)\cos 26, \vec{a}+\vec{b} = 22, \frac{\sin x}{44} = \frac{\sin 26}{22}, x = 61.3^\circ$